Final Exam — Functional Analysis (WIFA–08)

Monday 8 April 2019, 9.00–12.00h

University of Groningen

Instructions

1. The use of calculators, books, or notes is not allowed.
2. All answers need to be accompanied with an explanation or a calculation: only answering "yes", "no", or "42" is not sufficient.
3. If p is the number of marks then the exam grade is $G = 1 + p/10$.

Problem 1 (16 + 3 + 6 = 25 points)

Let S be any nonempty set and consider the normed linear space

$$\mathcal{B}(S,\mathbb{K}) = \left\{ f : S \to \mathbb{K} \,:\, \sup_{x\in S} |f(x)| < \infty \right\}, \quad \|f\|_\infty = \sup_{x\in S} |f(x)|.$$

(a) Prove that $(\mathcal{B}(S,\mathbb{K}), \|\cdot\|_\infty)$ is a Banach space.

In the following, assume that $S = \{x_n \,:\, n \in \mathbb{N}\}$ is infinite and countable.

(b) Show that $V = \{f \in \mathcal{B}(S,\mathbb{K}) \,:\, \sum_{n=1}^\infty |f(x_n)| < \infty\}$ is a linear subspace of $\mathcal{B}(S,\mathbb{K})$.

(c) Define on V the norm $\|f\|_1 = \sum_{n=1}^\infty |f(x_n)|$. Are the norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$ equivalent on V?

Problem 2 (3 + 3 + 8 + 8 + 3 = 25 points)

Provide the space $\mathcal{C}([0,1],\mathbb{R})$ with the sup-norm:

$$\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|.$$

Consider the linear operator $T : \mathcal{C}([0,1],\mathbb{R}) \to \mathcal{C}([0,1],\mathbb{R})$ defined by

$$Tf(x) = \int_0^1 K(x,t) f(t)\,dt \quad \text{where} \quad K(x,t) = \begin{cases} t(1-x) & \text{if } 0 \le t \le x \le 1, \\ x(1-t) & \text{if } 0 \le x \le t \le 1. \end{cases}$$

(a) Show that T is bounded. (Do not attempt to compute $\|T\|$!)

(b) Show that

$$Tf(x) = (1-x)\int_0^x t f(t)\,dt + x\int_x^1 (1-t) f(t)\,dt.$$

(c) Assume $\lambda \neq 0$. Prove the following implication:

$$Tf = \lambda f \quad \Rightarrow \quad f''(x) = -\frac{1}{\lambda} f(x), \quad f(0) = 0, \quad f(1) = 0.$$

(d) Show that $\lambda_n = 1/n^2\pi^2$ with $n \in \mathbb{N}$ is an eigenvalue of T and compute the corresponding eigenspace.

Hint: you may use without proof that the implication in part (c) is in fact an equivalence.

(e) Show that $0 \in \sigma(T)$.

Problem 3 (5 + (9 + 3 + 3) = 20 points)

(a) Formulate the Closed Graph Theorem.

(b) Let X be a Hilbert space, and assume that the sequence (h_n) in X satisfies

$$\sum_{n=1}^{\infty} |(x, h_n)|^2 < \infty \quad \text{for all} \quad x \in X.$$

In addition, consider the linear operator

$$T : X \to \ell^2, \quad Tx = ((x, h_1), (x, h_2), (x, h_3), \dots).$$

Prove the following statements:

(i) T is closed;

(ii) T is bounded;

(iii) There exists a constant $C > 0$ such that

$$\sum_{n=1}^{\infty} |(x, h_n)|^2 \leq C\|x\|^2 \quad \text{for all } x \in X.$$

Problem 4 (4 + 8 + 8 = 20 points)

(a) Formulate the Hahn-Banach Theorem for normed linear spaces.

(b) Consider the space $\mathcal{C}([0, 1], \mathbb{K})$ with the sup-norm. Fix $c \in [0, 1]$ and consider the following linear maps:

$$f : \mathcal{C}([0, 1], \mathbb{K}) \to \mathbb{K}, \qquad f(\varphi) = \int_0^1 \varphi(t)\, dt,$$

$$g : \mathcal{C}([0, 1], \mathbb{K}) \to \mathbb{K}, \qquad g(\varphi) = \varphi(c).$$

Show that $\|f\| = 1$ and $\|g\| = 1$.

(c) Consider the linear subspace $V = \text{span}\,\{1, x\}$ and the linear map

$$h : V \to \mathbb{K}, \qquad h(a + bx) = a + b/2.$$

Apply the Hahn-Banach Theorem to h: is the object of which the existence is asserted by that theorem unique?

End of test (90 points)

Solution of Problem 1 (16 + 3 + 6 = 25 points)

(a) Let (f_n) be a Cauchy sequence in $\mathcal{B}(S,\mathbb{K})$. Then for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$n, m \geq N \quad \Rightarrow \quad \|f_n - f_m\|_\infty \leq \varepsilon.$$

Let $x \in S$ be arbitrary. Then

$$n, m \geq N \quad \Rightarrow \quad |f_n(x) - f_m(x)| \leq \sup_{x \in S} |f_n(x) - f_m(x)| = \|f_n - f_m\|_\infty \leq \varepsilon, \tag{1}$$

which shows that $f_n(x)$ is a Cauchy sequence in $\mathbb{K}$. Since $\mathbb{K}$ is complete $\lim_{n\to\infty} f_n(x)$ exists. Hence, we can define a function $f : S \to \mathbb{K}$ pointwise by setting $f(x) := \lim_{n\to\infty} f_n(x)$.
(7 points)

By letting $m \to \infty$ in equation (1) we obtain that

$$n \geq N \quad \Rightarrow \quad |f_n(x) - f(x)| \leq \varepsilon.$$

Since this inequality holds for all $x \in S$, we obtain

$$n \geq N \quad \Rightarrow \quad \|f_n - f\|_\infty \leq \varepsilon, \tag{2}$$

which means that $f_n \to f$ in $\mathcal{B}(S,\mathbb{K})$.
(7 points)

In particular, equation (2) also implies that $f_N - f \in \mathcal{B}(S,\mathbb{K})$. Since $\mathcal{B}(S,\mathbb{K})$ is a linear space it follows that $f = f_N - (f_N - f) \in \mathcal{B}(S,\mathbb{K})$.
(2 points)

(b) If $f, g \in V$ and $\lambda \in \mathbb{K}$, then

$$\sum_{n=1}^\infty |f(x_n) + g(x_n)| \leq \sum_{n=1}^\infty \big(|f(x_n)| + |g(x_n)|\big) = \sum_{n=1}^\infty |f(x_n)| + \sum_{n=1}^\infty |g(x_n)| < \infty$$

and

$$\sum_{n=1}^\infty |\lambda f(x_n)| = |\lambda| \sum_{n=1}^\infty |f(x_n)| < \infty,$$

which implies that $f + g \in V$ and $\lambda f \in V$ as well.
(3 points)

(c) Define for $n \in \mathbb{N}$ the following function:

$$f_n : S \to \mathbb{K}, \qquad f_n(x_k) = \begin{cases} 1 & \text{if } k \leq n, \\ 0 & \text{if } k > n. \end{cases}$$

On the one hand, we have that $\|f_n\|_1 = n$ for all $n \in \mathbb{N}$ (which also shows that $f_n \in V$). On the other hand $\|f_n\|_\infty = 1$ for all $n \in \mathbb{N}$. Therefore, the norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are not equivalent on V.
(6 points)

Solution of Problem 2 (3 + 3 + 8 + 8 + 3 = 25 points)

(a) Observe that $|K(x,t)| \leq 1$ for all $(x,t) \in [0,1] \times [0,1]$. Therefore, it follows for all $x \in [0,1]$ that

$$|Tf(x)| = \left| \int_0^1 K(x,t)f(t)\,dt \right| \leq \int_0^1 |K(x,t)|\,|f(t)|\,dt \leq \int_0^1 \|f\|_\infty\,dt = \|f\|_\infty.$$

Hence,

$$\|Tf\|_\infty = \sup_{x\in[0,1]} |Tf(x)| \leq \|f\|_\infty,$$

which shows that T is indeed bounded.
(3 points)

(b) We have that

$$\begin{aligned} Tf(x) &= \int_0^1 K(x,t)f(t)\,dt \\ &= \int_0^x K(x,t)f(t)\,dt + \int_x^1 K(x,t)f(t)\,dt \\ &= \int_0^x t(1-x)f(t)\,dt + \int_x^1 x(1-t)f(t)\,dt \\ &= (1-x)\int_0^x tf(t)\,dt + x\int_x^1 (1-t)f(t)\,dt. \end{aligned}$$

(3 points)

(c) If $Tf = \lambda f$, then

$$\begin{aligned} \lambda f(x) &= (1-x)\int_0^x tf(t)\,dt + x\int_x^1 (1-t)f(t)\,dt \\ &= (1-x)\int_0^x tf(t)\,dt - x\int_1^x (1-t)f(t)\,dt. \end{aligned}$$

Since $\lambda \neq 0$ it immediately follows that $f(0) = f(1) = 0$.
(2 points)

Differentiation gives

$$\begin{aligned} \lambda f'(x) &= -\int_0^x tf(t)\,dt + (1-x)xf(x) - \int_1^x (1-t)f(t)\,dt - x(1-x)f(x) \\ &= -\int_0^x tf(t)\,dt - \int_1^x (1-t)f(t)\,dt. \end{aligned}$$

(4 points)

Differentiating again gives

$$\lambda f''(x) = -xf(x) - (1-x)f(x) = -f(x).$$

(2 points)

(d) For $\lambda_n = 1/n^2\pi^2$ we obtain the boundary value problem:

$$f''(x) = -n^2\pi^2 f(x), \quad f(0) = 0, \quad f(1) = 0.$$

The general solution of the differential equation is given by

$$f(x) = A\cos(n\pi x) + B\sin(n\pi x).$$

(4 points)

The boundary condition $f(0) = 0$ implies that $A = 0$.
(1 point)

The boundary condition $f(1) = 0$ is satisfied for any constant B.
(1 point)

Hence, the eigenspace of T corresponding to the eigenvalue $\lambda_n = 1/n^2\pi^2$ is given by $\ker\,(T - \lambda_n) = \operatorname{span}\{\sin(n\pi x)\}$.
(2 points)

(e) Since $\lambda_n \in \sigma(T)$ for all $n \in \mathbb{N}$ and $\lambda_n \to 0$ it follows that $0 \in \operatorname{clos}\sigma(T) = \sigma(T)$.
(3 points)

Solution of Problem 3 (5 + (9 + 3 + 3) = 20 points)

(a) Let X and Y be Banach spaces, let $V \subset X$ be a closed linear subspace, and let $T : V \to Y$ be a linear map. If the graph of T is closed, then $T \in B(V, Y)$.
(5 points)

(b) (i) Assume that $x_n \to x$ in X and $Tx_n \to y$ in ℓ^2. Then $(x_n, Tx_n) \to (x, y)$ in the product space $X \times \ell^2$. We need to show that $(x, y) \in G(T)$, or, equivalently, $y = Tx$.

Write $y = (y_1, y_2, y_3, \dots)$. Note that the subscripts in x and y have different meanings. One the one hand, we have for each fixed $k \in \mathbb{N}$ that

$$|(x_n, h_k) - y_k| \leq \sqrt{\sum_{k=1}^{\infty} |(x_n, h_k) - y_k|^2} = \|Tx_n - y\| \to 0 \quad \text{as} \quad n \to \infty.$$

(3 points)

On the other hand we have

$$|(x_n, h_k) - (x, h_k)| = |(x_n - x, h_k)| \leq \|x_n - x\| \, \|h_k\| \to 0 \quad \text{as} \quad n \to \infty.$$

(3 points)

By uniqueness of limits it follows that $y_k = (x, h_k)$ for all $k \in \mathbb{N}$, which implies that $y = Tx$. We conclude that T is a closed operator.
(3 points)

(ii) Applying the Closed Graph Theorem with the Hilbert spaces X and $Y = \ell^2$ and the closed linear subspace $V = X$ gives that T is a bounded operator.
(3 points)

(iii) Since T is bounded we have $\|Tx\|^2 \leq \|T\| \, \|x\|$ for all $x \in X$, or, equivalently,

$$\sum_{n=1}^{\infty} |(x, h_n)|^2 \leq \|T\|^2 \, \|x\|^2,$$

which means that we can take $C = \|T\|^2$.
(3 points)

Solution of Problem 4 (4 + 8 + 8 = 20 points)

(a) Let X be a normed linear space and let $V \subset X$ be a linear subspace. If $f \in V'$, then there exists $F \in X'$ such that $F(v) = f(v)$ for all $v \in V$ and $\|F\| = \|f\|$. **(4 points)**

(b) For $\varphi \in \mathcal{C}([0,1], \mathbb{K})$ we have that

$$|f(\varphi)| = \left| \int_0^1 \varphi(t)\,dt \right| \leq \int_0^1 |\varphi(t)|\,dt \leq \int_0^1 \|\varphi\|_\infty\,dt = \|\varphi\|_\infty.$$

(3 points)

With the function $\varphi(t) = 1$ we have $\|\varphi\|_\infty = 1$ and $|f(\varphi)| = 1$. Hence,

$$\|f\| = \sup_{\varphi \neq 0} \frac{|f(\varphi)|}{\|\varphi\|_\infty} = 1.$$

(1 point)

For $\varphi \in \mathcal{C}([0,1], \mathbb{K})$ we have that

$$|g(\varphi)| = |\varphi(c)| \leq \sup_{x \in [0,1]} |\varphi(x)| = \|\varphi\|_\infty.$$

(3 points)

With the function $\varphi(t) = 1$ we have $\|\varphi\|_\infty = 1$ and $|g(\varphi)| = 1$. Hence,

$$\|g\| = \sup_{\varphi \neq 0} \frac{|g(\varphi)|}{\|\varphi\|_\infty} = 1.$$

(1 point)

(c) First observe that with $c = \frac{1}{2}$ it follows that $f(\varphi) = g(\varphi) = h(\varphi)$ for all $\varphi \in V$. **(1 point)**

In particular, it then follows that

$$\|h\| = \sup_{\varphi \in V \setminus \{0\}} \frac{|h(\varphi)|}{\|\varphi\|_\infty} = \sup_{\varphi \in V \setminus \{0\}} \frac{|f(\varphi)|}{\|\varphi\|_\infty} \leq \sup_{\varphi \in \mathcal{C}([0,1],\mathbb{K}) \setminus \{0\}} \frac{|f(\varphi)|}{\|\varphi\|_\infty} = \|f\| = 1.$$

But note that with $\varphi(t) = 1$ we have $\|\varphi\|_\infty = 1$ and $|h(\varphi)| = 1$, which implies that $\|h\| = 1$.
(3 points)

We conclude that both f and g with $c = \frac{1}{2}$ are norm preserving extensions of h. But note that $f \neq g$, since for $\varphi(t) = t^2$ we have $f(\varphi) = \frac{1}{3}$ whereas $g(\varphi) = \frac{1}{4}$. Therefore, the norm preserving extension of h obtained by the Hahn-Banach Theorem is not unique.
(4 points)